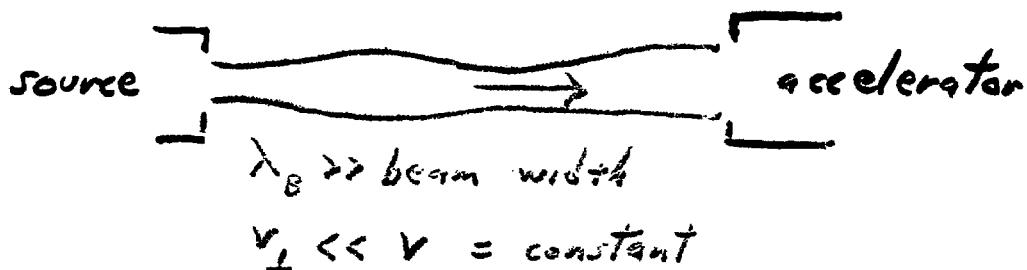


Emittance Growth in Low Energy Beam Transport

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Low energy beam transport (LEBT):

Move DC beam under influence of space charge and external focusing forces from source to accelerator:



Classic problem in LEBT: Try to maintain low emittance of beam exiting source, but emittance increases irreversibly over a distance of order λ_B if density profile is nonuniform.

Description using single particle orbits:

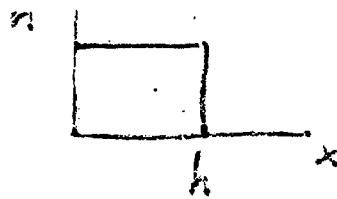
$$\frac{d^2x}{dz^2} = x'' = \frac{F_e + F_s}{m\gamma v^2}$$

focusing: $\frac{F_e}{m\gamma v^2} = -k^2(z)x = -k^2x$
smooth approximation

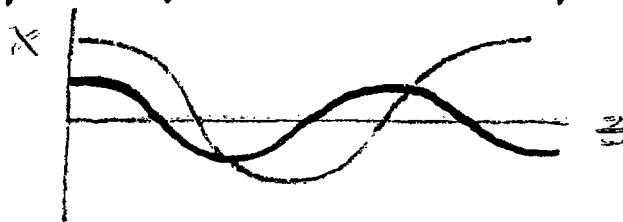
space charge: $\frac{F_s}{m\gamma v^2} = \frac{e}{m\gamma v^2} \left(E_{sx} - \frac{v}{c} B_{sy} \right)$

$$= \frac{e E_{sx}(x)}{m\gamma^3 v^2} \rightarrow 0 \text{ as } v \rightarrow c$$

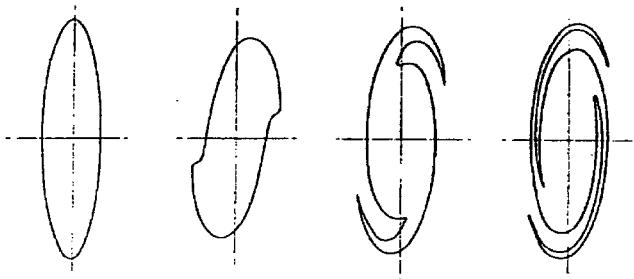
$$x'' = -k^2 x + \frac{e E_{sx}(x)}{m \gamma^3 v^2}$$



If density profile is uniform, $E_{sx} \sim x$ and motion is harmonic. Otherwise oscillation frequency depends on amplitude:



"Dilution" of emittance



Beam moment approach to emittance growth:

envelope equation: $R'' + k^2 R - \frac{\epsilon^2}{R^3} - \frac{K}{R} = 0$

$$R = \langle r^2 \rangle^{1/2} = \text{rms beam size}$$

$$K = \frac{\langle r F_s \rangle}{m \gamma^3 v^2} = \frac{2 N e^2}{m \gamma^3 v^2} = \text{perveance}$$

$$N = \# \text{ particles / length}$$

$$\epsilon^2 = 16 (\langle r^2 \rangle \langle r'^2 \rangle - \langle rr' \rangle^2)$$

Lapostolle & Sacherer independently (1971) realized that space charge field energy of beam induces changing rms emittance (potential \leftrightarrow kinetic energy).

$$\frac{d\epsilon^2}{dz} = \frac{32}{mv^2} (\langle r^2 \rangle \langle r' F_s \rangle - \langle rr' \rangle \langle r F_s \rangle)$$

$$= -K \langle r^2 \rangle \frac{d}{dz} U_n$$

$$U_n = \frac{\int E_s^2 r dr - \int E_s^2 (\text{uniform}) r dr}{\int E_s^2 (\text{uniform}) r dr}$$

"free space
charge
field energy"

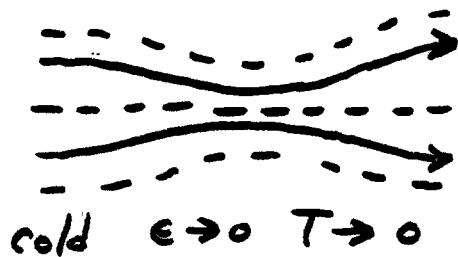
How does emittance evolve in transport lines?

Cannot calculate $\langle r' F_s \rangle$ self-consistently since F_s is usually nonlinear function of r .

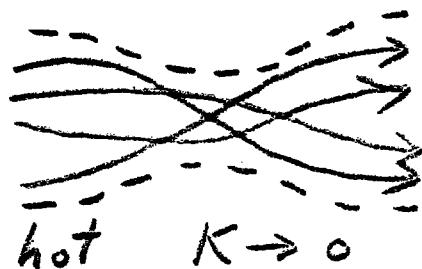
In LEBT, beam is usually "space charge dominated":

$$\frac{K}{R} \gg \frac{\epsilon^2}{R^3} \quad \text{or} \quad \frac{KR^2}{\epsilon^2} \gg 1$$

Anderson (1985): Space charge dominated beam acts like a "cold" beam ($\epsilon \rightarrow 0$):



cold $\epsilon \rightarrow 0$ $T \rightarrow 0$



hot $K \rightarrow 0$

Particle trajectories follow laminar flow i.e. x' is single-valued function of x

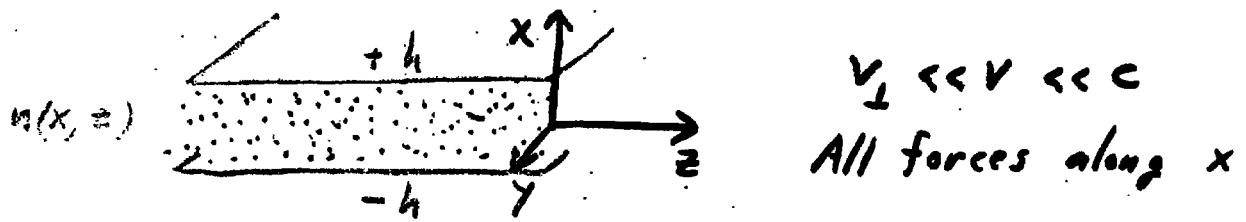
Trajectories cross.

Debye length:

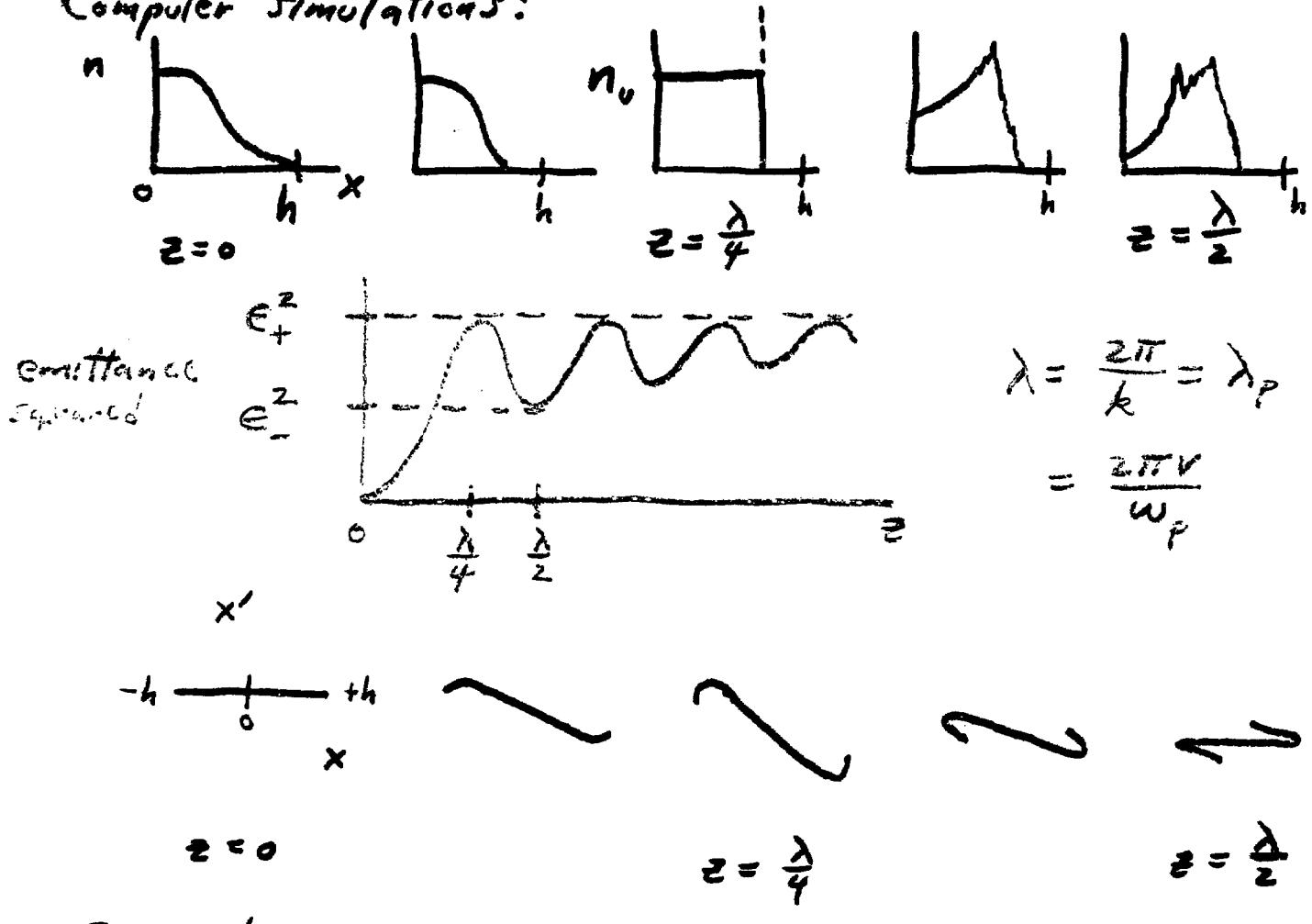
$$\frac{\lambda_D^2}{a^2} = \langle v_\perp^2 \rangle / w_p^2 a^2$$

$$= \frac{1}{a^2} \frac{\epsilon^2}{N e^2 / m \sigma v^2}$$

Consider cold 1-d (sheet) beam moving at constant velocity v along z axis in uniform focussing channel.



Computer simulations:



$$\begin{aligned}\lambda &= \frac{2\pi}{k} = \lambda_p \\ &= \frac{2\pi v}{\omega_p}\end{aligned}$$

Space charge potential energy \rightarrow kinetic energy "shock" \rightarrow thermal energy

Emittance = fluid part (reversible) + thermal part



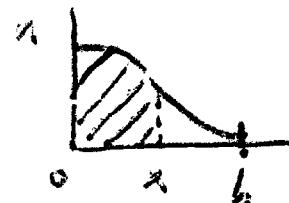
Anderson (LBL 1986):

For cold beam, motion is initially laminar.

Emittance growth can be calculated up to initial shock.

$$N_x(x, z) = \int_0^x n(x_i, z) dx_i$$

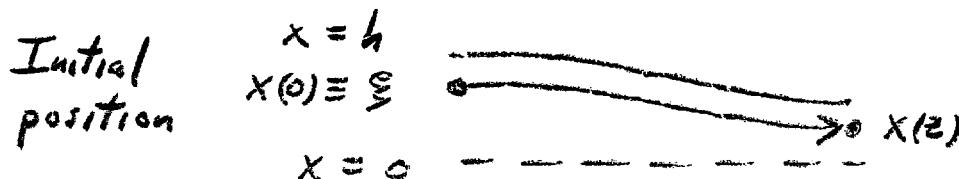
$$N = \int_0^\infty n(x, z) dx$$



$$n = \partial N_x / \partial x \quad \text{Poisson: } \frac{\partial N_x}{\partial x} \approx \frac{1}{4\pi e} \frac{\partial}{\partial x} E_s(x, z)$$

$$\text{Space charge } E_s = 4\pi e N_x$$

Laminar motion $\rightarrow N_x$ preserved for each particle at its position $x(z)$



$$N_x(x, z) = N_x(\xi, 0)$$

In uniform focussing channel: $\frac{F_e}{mv^2} \equiv -k^2 x$

$$\frac{d^2 x}{dz^2} = -k^2 x + \frac{eE_s}{mv^2} = -k^2 \left(x - \underbrace{\frac{P}{k^2} \frac{N_x}{N}}_{x_e(\xi)} \right)$$

$$P = \frac{4\pi Ne^2}{mv^2} = \text{"perveance"}$$

Laminar motion:

$$x(\xi, z) = x_e(\xi) + (\xi - x_e(\xi)) \cos kz$$

Laminar motion ceases when two beam elements separated initially by $d\xi$ are later separated by $dx = 0$. i.e. $\frac{dx(\xi, z)}{d\xi} = 0$ (trajectory crossing)

$$\cos k z_c = \frac{1}{1 - \frac{n_0}{n(\xi_c)}}$$

$$n_0 = \frac{N}{P/k^2} \quad (\text{cold uniform matched beam density})$$

ξ_c is point of minimum initial density

Laminarity Criterion:

If $n(\xi) > n_0/2$ for all $\xi < h$,

z_c does not exist and motion is forever laminar.

* If $n(\xi) \leq n_0/2$ for some ξ , that part of beam with minimum density $n(\xi_c)$ will cross trajectories first.

* If z_c exists, then $\frac{\lambda}{4} \leq z_c \leq \frac{\lambda}{2}$

For laminar motion, $n(x, z) dx = n(\xi) d\xi$

$$n(x, z) = \frac{n_0}{1 + \left(\frac{n_0}{n(\xi)} - 1\right) \cos kz}$$

If laminarity is violated at $z = z_c$, then $n \rightarrow \infty$ for particles originating at $x = \xi_c$.

Rms emittance during laminar motion:

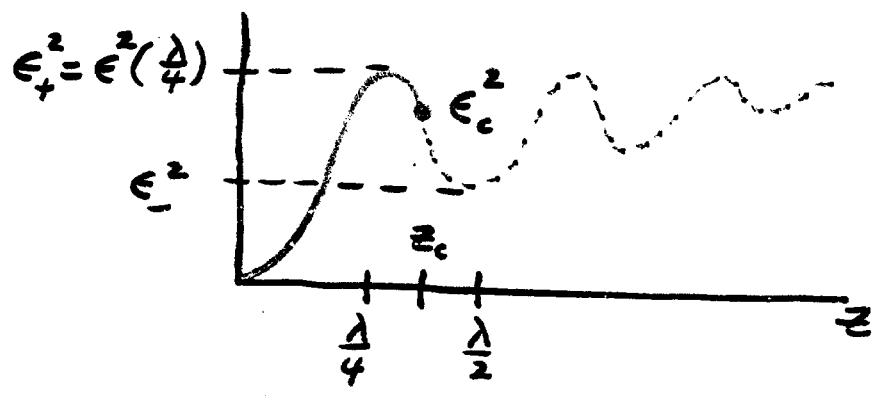
$$\epsilon^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$

Use $\langle g \rangle(z) = \frac{1}{N} \int_0^\infty dx n(x, z) g(x) = \frac{1}{N} \int_0^\infty d\xi n(\xi) g(x(\xi, z))$

$$\epsilon^2(z) = \frac{P^2}{k^2} \left[\frac{x_c^2}{3} - \frac{w_0^2}{P^2} \right] \sin^2 k z \quad (\text{Anderson})$$

$$x_c^2 = \langle x^2 \rangle(z=0) \quad (\text{mean square beam half width})$$

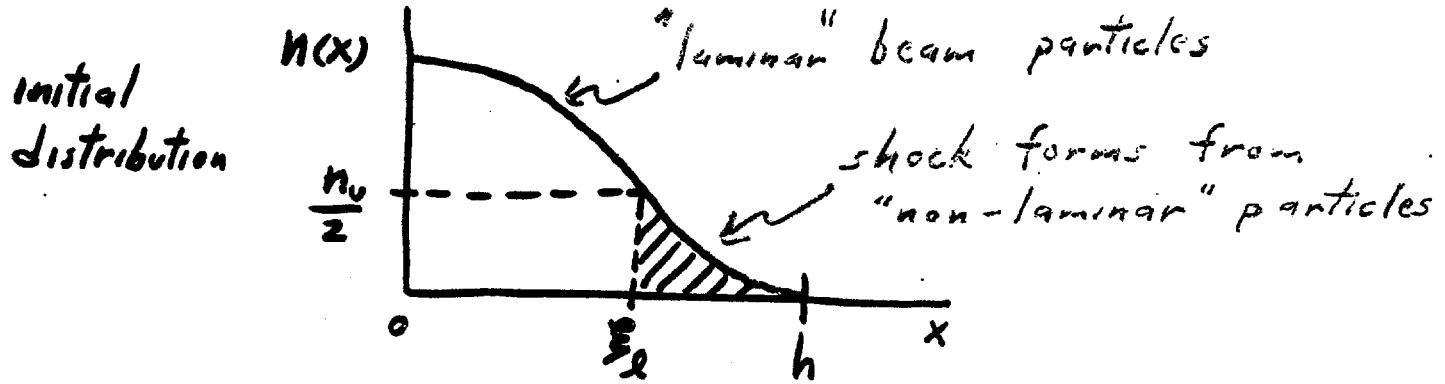
$$w_0 = W(z=0) = \frac{P}{N} \langle N_x x \rangle_{z=0} \quad (\text{"visual moment"})$$



The behavior for $z > z_c$ is the result of shock-like, quasi-thermal particle motion and not immediately calculable.

Seek empirical formula for ϵ_-^2 by computer simulation of matched sheet beams.

The emittance ϵ_-^2 probably depends strongly on quantities relevant to the shock region of the beam where $n(\xi) \leq n_0/2$.



$$\text{Matched beam : } X_0^2 = \frac{1}{3} \left(\frac{P}{k} \right)^2$$

$$n_0 = \frac{N}{P/k^2} = \frac{N}{\sqrt{3} X_0}$$

z_0 is the boundary between "laminar" and "non-laminar" particles of initial distribution.

Simulation results (6000 particles, 300 grid points) :

For initially peaked beams with monotonically decreasing density, one finds the remarkably simple relation

$$\frac{\epsilon^2}{\epsilon_c^2} \approx \left(1 - \left(\frac{z_0}{h} \right)^{7/4} \right) \left(2 - \frac{z_0}{\lambda/4} \right)^{6/5} \quad (\text{accurate to } 10\%)$$

This presumably mimics some as yet uncalculated formula.

